On Black Hole Horizon Fluctuations

K.L. Tuchin[1]

Raymond and Beverly Sackler Faculty of Exact Science,
School of Physics and Astronomy,
Tel-Aviv University, Ramat Aviv, 69978, Israel

February 7, 2008

Abstract

A study of the high angular momentum particles 'atmosphere' near the Schwarzschild black hole horizon suggested that strong gravitational interactions occur at invariant distance of the order of $\sqrt[3]{M}$. We present a generalization of this result to the Kerr-Newman black hole case. It is shown that the larger charge and angular momentum black hole bears, the larger invariant distance at which strong gravitational interactions occur becomes. This invariant distance is of order $\sqrt[3]{r_+^2/(r_+ - r_-)}$. This implies, that the Planckian structure of the Hawking radiation of extreme black holes is completely broken.

PACS: 04.60.-m; 04.70Dy
Keywords: black hole, atmosphere, fluctuations, semi-classical approximation

1 Introduction

The black hole radiation is a direct consequence of non-static character of the collapsing body metric. A wave propagated through the collapsing body becomes red-shifted, hence an in-vacuum state defined with respect to modes at past null infinity differs from an out-vacuum state, defined with respect to modes at future null infinity. The red-shift is exponentially large if collapse leads to the black hole formation.

It was shown by Hawking[1] that the black hole radiation spectrum is the Planckian one, provided that there is no back reaction of emerging quanta on metric; the matter fields propagate on a classical background metric, and the wave equation is valid at all scales.

This semi-classical approach of Hawking becomes doubtful at scales of the order of $l_p = \sqrt{G\hbar/c^3}$, i.e. the ultraviolet part $\omega > l_p^{-1}$ of the Hawking radiation spectrum is possibly non-Planckian. However, such huge energies are of little interest from an experimentalist point of view. It turns out that problems begin at much smaller scale.

[1]e-mail:tuchin@post.tau.ac.il

We can gain more by putting in question the possibility to neglect the radiation back reaction. Existence of energy flux from the black hole at infinity implies that the black hole mass decreases. However, as long as the mass of the black hole is large compared to the Planck mass, the rate of evolution of the black hole is small compared to the characteristic time for the light to cross the gravitational radius[8]. Thus, we can describe the black hole by a sequence of stationary solutions; in each solution the back reaction influence on the radiation spectrum can be neglected[1].

On the other hand, despite the fact that the Riemann tensor is small near the black hole horizon $\mathbf{R} \sim 1/M^2$ the normal modes get exponentially red-shifted[3]. This implies, that strong gravitational interactions occur there and, possibly, alter the Planckian character of the spectrum.

The brick wall model proposed by 't Hooft is a simple model that provides some insight into the problem[4]. The idea is to calculate the number of the (scalar) wave equation solutions near the horizon of the Schwarzschild black hole. This number can be interpreted as a partition function of the black hole "atmosphere" — a gas of high angular momentum Hawking particles reflected back by the space-time curvature[2]. The atmosphere's entropy turns out to be divergent at the horizon. However, one can introduce an invariant cut-off ρ of the order of 1 and independent of M, such that the atmosphere's entropy becomes exactly the Bekenstein-Hawking one. This implies that the strong gravitational interactions occur at invariant distance of the order of 1. Several other authors arrive at this conclusion using other models[3].

However, some authors arrive at opinion that ρ is greater that 1[3]. In particular, Casher *et. al.* showed that ρ is of the order $M^{1/3}$, provided that one takes account of the atmosphere's thermal fluctuations. This implies that the Hawking radiation Planckian structure is broken at $\omega > M^{-1/3}$. Therefore, unlike the Planckian spectrum quanta, the real black hole radiation quanta are correlated, and information about the state of in-falling matter gets encoded out of the horizon[2].

The brick wall model may be criticized for it deals with non-renormalized stress tensor. We can avoid this difficulty as discussed in ref. [2]. Both approaches yield the same result for the atmosphere's entropy. So, we shall apply the brick wall model again in order to obtain an expression for the charged and rotating black hole entropy (sec. 2 and 2.3)[11].

Along the way we shall learn more about the atmosphere. In particular, it will be shown that the main contribution to the partition function comes from the particles emitted with a charge and angular momentum of the same sign as the black hole ones.

In section 2.2 the atmosphere thermal fluctuations will be studied. It will be shown that strong gravitational interactions near the charge black hole horizon occur at an invariant distance of the order of $[r_+^2/(r_+ - r_-)]^{1/3}$. In section 2.3 it will be argued that this result is valid for the Kerr black hole also.

In section 3 we shall confirm result of section 2.2 by applying the shock wave model[12] to study the gravitational interactions between in-falling and out-going particles.

We discuss the results in section 4.

2 Thermal Properties of Atmosphere

2.1 Charged Black Holes

The charged black hole metric (Reissner-Nordstrøm geometry) is given by:

$$ds^2 = \frac{D}{r^2}dt^2 - \frac{r^2}{D}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where $D = r^2 - 2Mr + Q^2$, M and Q are the mass and the charge of black hole respectively. The event horizon is located at surface r_+ defined as

$$r_{\substack{+ \\ -}} = M \substack{+ \\ -} \sqrt{M^2 - Q^2}. \tag{2}$$

A distant inertial observer observes the Hawking radiation — energy flux from the black hole. This flux has a Planckian structure provided that we neglect the back reaction of this radiation on the metric[1]. Therefore, the Hawking radiation can be considered as a black body radiation in a thermal bath with the temperature T_H

$$T_H = \frac{1}{8\pi M}(1 - \frac{Q^4}{r_+^4}) = \frac{r_+ - r_-}{\mathcal{A}}, \tag{3}$$

where $\mathcal{A} = 4\pi r_+^2$ is the horizon area[8].

Consider a massless neutral scalar field ϕ in the charged black hole background. One can separate variables in the wave equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$$

and obtain the following expressions for the basis functions[9]:

$$u_{Elm}(x) = r^{-1}R_l(r)Y_{lm}(\theta,\varphi)e^{-iEt},$$

where $l = 0, 1, 2\ldots$, $m = -l, -l+1, \ldots, l-1, l$, $Y_{lm}(\theta,\varphi)$ are the spherical harmonics, E is the hamiltonian eigenvalue. The radial part $R_l(r)$ satisfies the following equation:

$$[\frac{d^2}{dr^{*2}} - V_{El}(r)]R_l(r) = 0, \tag{4}$$

where r^* and $V_{El}(r)$ are defined by the following relations:

$$\frac{dr^*}{dr} = \frac{r^2}{D}, \tag{5}$$

and

$$V_{El}(r) = -E^2 + l^2\frac{D}{r^4} + \frac{2MrD}{r^6}. \tag{6}$$

The centrifugal barrier is

$$V_l^c(r) = l^2 \frac{D}{r^4}.$$

It is attractive for $r_+ \leq r < r_1$ and repulsive for $r > r_1$, where r_1 is the root of the equation $\frac{dV^c}{dr} = 0$:

$$r_1 = \frac{3}{2}\left(M + \sqrt{M^2 - \frac{8}{9}Q^2}\right), \tag{7}$$

The tunneling through the angular momentum barrier may be neglected for all but the lowest angular momentum modes since $V_l^c \sim l^2$[3].

We see that near the horizon there exists an atmosphere of the neutral high angular momentum particles. In general, the same is also true for the *charged* scalar particles which are defined with respect to basis functions of the following equation:

$$g^{\mu\nu}(\nabla_\mu - ieA_\mu)(\nabla_\nu - ieA_\nu)\phi. \tag{8}$$

Let us study properties of this thermodynamical system. At first we shall calculate the number of the wave equation solutions Γ. As long as $M \gg 1$ we can rely on WKB approximation. So we get:

$$\Gamma = (2\pi)^{-3} \sum_g \int dx^k dp_k; \quad k = r, \theta, \varphi, \tag{9}$$

where the sum runs over all degenerate energy levels.

The covariant radial momentum p_r associated with the differential operator $\hat{p}_r$ in the wave equation is given by[6, 7]:

$$p_r^2 = D^{-2}(r^2 E - eQr)^2 - D^{-1} l^2. \tag{10}$$

Here e is the charge of the scalar particles. Since the metric is spherically symmetric integration over the θ degree of freedom in eq. (9) is trivial. Thus,

$$\Gamma(E) = \sum_l (2\pi)^{-2} \sum_m \int d\varphi \int dr \int dp_r \approx \frac{1}{2\pi} \int dl(2l+1) \int dr \int dE \frac{dp_r}{dE}.$$

This expression is divergent at the horizon and at infinity. To proceed further we should introduce cut-offs. To this end let us use the brick wall model [4].

We introduce the following boundary conditions: $\phi(x) = 0$ if $r \geq L$, and $\phi(x) = 0$ if $r \leq r_+ + h$, where L and h are the infrared and ultraviolet cutoffs respectively. We are interested in the contribution of the horizon, i.e. in the term $O(h^{-1})$. The term proportional to L^3 is the usual contribution from the vacuum surrounding the system at large distances and is of little relevance here. So we find:

$$D \approx (r_+ - r_-)h, \tag{11}$$

$$\Gamma(E) \approx \frac{1}{\pi} \int dE \int_0^{l_m} dl\, l \frac{r_+^4 (E - e\Phi) D^{-2} h}{\sqrt{D^{-2} r_+^4 (E - e\Phi)^2 - D^{-1} l^2}}$$

$$= \frac{1}{\pi h} \frac{r_+^6}{(r_+ - r_-)^2} \int dE (E - e\Phi)^2, \tag{12}$$

where l_m is the largest possible value of l which arises from the obvious condition $p_r^2 \geq 0$:

$$l^2 \leq \frac{r_+^4 (E - e\Phi)^2}{h(r_+ - r_-)^2},$$

and $\Phi = Q/r_+$ is the electric potential at the horizon.

The free energy of the atmosphere is given by:

$$F = T_H \int d\Gamma(E) \ln(1 - e^{(e\Phi - E)\beta}). \tag{13}$$

This integral exists only if the condition $e\Phi - E \leq 0$ holds, i.e. the supperradiant modes do not contribute to the atmosphere's free energy. Hence,

$$\begin{aligned}
\Gamma(E) &= \frac{1}{\pi h} \frac{r_+^6}{(r_+ - r_-)^2} \times \\
&\times \left[\int_0^E (E' - e\Phi)^2 dE' \theta(-eQ) + \int_{e\Phi}^E (E' - e\Phi)^2 dE' \theta(eQ) \right] \\
&= \frac{1}{3\pi h} \frac{r_+^6}{(r_+ - r_-)^2} \times \\
&\times \left[\{(E - e\Phi)^3 + (e\Phi)^3\} \theta(-eQ) + \{(E - e\Phi)^3\} \theta(eQ) \right] \\
&\equiv \Gamma(E)_- + \Gamma(E)_+
\end{aligned}$$

and also

$$\begin{aligned}
F &= \left[T_H \Gamma(E) \ln(1 - e^{(e\Phi - E)\beta})|_{E=0}^{\infty} - \int_0^\infty dE \frac{\Gamma(E)}{e^{(E - e\Phi)\beta} - 1} \right] \theta(-eQ) \\
&+ \left[T_H \Gamma(E) \ln(1 - e^{(e\Phi - E)\beta})|_{E=e\Phi}^{\infty} - \int_{e\Phi}^\infty dE \frac{\Gamma(E)_-}{e^{(E - e\Phi)\beta} - 1} \right] \theta(eQ) \\
&= - \int_0^\infty dE \frac{\Gamma(E)}{e^{(E - e\Phi)\beta} - 1} \theta(-eQ) - \int_{e\Phi}^\infty dE \frac{\Gamma(E)_+}{e^{(E - e\Phi)\beta} - 1} \theta(eQ) \\
&\equiv F_- + F_+,
\end{aligned}$$

where we have assigned the subscript $+$ to the contribution of particles with the same sign of charge as the black hole's one, and the subscript $-$ to the contribution of their oppositely charged antiparticles. The step-function $\theta(x)$ is defined as usual: $\theta(x) = 1$, if $x \geq 0$, and $\theta(x) = 0$, if $x \leq 0$. Substitution of Γ yields:

$$F_- = - \frac{r_+^6}{3\pi h (r_+ - r_-)^2} \int_0^\infty dE \frac{(E - e\Phi)^3 + (e\Phi)^3}{e^{(E - e\Phi)\beta} - 1}, \quad eQ \leq 0; \tag{14}$$

$$F_+ = - \frac{r_+^6}{3\pi h (r_+ - r_-)^2} \int_{e\Phi}^\infty dE \frac{(E - e\Phi)^3}{e^{(e\Phi - E)\beta} - 1}, \quad eQ \geq 0. \tag{15}$$

As long as we assume that $M \gg 1$, condition $\beta \gg 1$ holds for the black hole of any charge. Then neglecting -1 in the integrand denominator and changing the variable of integration $E - e\Phi = y$ one gets:

$$F_- = -\frac{r_+^6}{3\pi h(r_+ - r_-)^2}\left(T_H e^{e\beta\Phi}(e\Phi)^3 + \int_{-e\Phi}^{\infty} y^3 e^{-y\beta} dy\right)$$
$$= -\frac{r_+^6}{3\pi h(r_+ - r_-)^2} e^{e\beta\Phi} T_H^4 \left(3(e\beta\Phi)^2 - 6e\beta\Phi + 6\right), \quad eQ \leq 0; \tag{16}$$

$$F_+ = -\frac{r_+^6}{3\pi h(r_+ - r_-)^2}\int_0^{\infty} y^3 e^{-y\beta} dy = -\frac{r_+^6}{3\pi h(r_+ - r_-)^2} 6T_H^4, \quad eQ \geq 0. \tag{17}$$

From eqs.(16,17) we deduce that the contribution of particles carrying a charge with sign opposite to the black hole's one is negligible compared to the contribution of their antiparticles. This is because they are pulled into the black hole by the electrostatic field as soon as they emerge and thus spend only a short time outside the horizon. Conversely, particles with the same sign of charge as the black hole's one are pushed out of the horizon, then scatter off the effective potential V_l (see eq.(6)) and then the lowest angular momentum ones escape to infinity, and the others return to the atmosphere.

Note that F_+ does not depend on e. Thus, the thermal properties of the charged atmosphere (and, hence, contribution of the horizon to these properties) are the same as the neutral one and defined by the black hole mass and charge completely.

In the limit of the neutral scalar field ($e = 0$) eqs. (17,16) reduce to

$$F_n = F_-(e = 0) = F_+ = -\frac{2r_+^6}{\pi h(r_+ - r_-)^2} T_H^4. \tag{18}$$

The entropy of the neutral atmosphere is given by:

$$S_n = -\frac{\partial F_n}{\partial T_H} = \frac{8(r_+ - r_-)}{(4\pi)^3 \pi h}. \tag{19}$$

This coincides with the Hawking-Bekenstein entropy $S = \frac{1}{4}\mathcal{A}$ [1, 5] for the following value of the cutoff:

$$h = h_n = \frac{8(r_+ - r_-)}{(4\pi)^3 \pi^2 r_+^2}. \tag{20}$$

The fact that h depends upon M and Q is merely due to the special choice of coordinates. Define the invariant distance as follows:

$$\rho = \int_{r_+}^{r=r_++h} ds = \int_{r_+}^{r=r_++h} \sqrt{-g_{rr}} dr \approx \frac{2r_+}{(r_+ - r_-)^{1/2}}\sqrt{h}. \tag{21}$$

So, $\rho_n \equiv \rho(h_n) = \pi^{-5/2} 2^{-1/2}$.

We see that ρ is a property of the horizon independent of M and Q. The same result for the Schwarzschild black hole was obtained by 't Hooft[4].

In terms of the invariant distance the neutral atmosphere entropy reads:

$$S_n = \frac{32}{(4\pi)^3\pi} \cdot \frac{r_+^2}{\rho_n^2}.$$

Suppose now that $e \neq 0$. In any realistic black hole the following inequality holds: $|eQ|\beta \gg 1$. Indeed,

$$|eQ|\beta \sim |eQ|\frac{M^2}{(r_+ - r_-)} \sim |eQ|\frac{M}{\sqrt{1-(Q/M)^2}} \gg 1;$$

which is clearly true since $Q \gg e_{electron} \sim 10^{-1}$ and $M \gg 1$. Hence, in the leading order eq. (16) gives:

$$F_- = -\frac{r_+^6}{\pi h(r_+ - r_-)^2} T_H^2 e^{e\beta\Phi}(e\Phi)^2.$$

Since F_+ does not depend on e the entropy of the charged atmosphere S_c has only an exponentially small dependence on e:

$$\begin{aligned} S_c &\equiv S_+ + S_- \\ &\approx \frac{32}{(4\pi)^4\pi} \cdot \frac{r_+^2}{\rho^2} + \frac{r_+^6}{4\pi h} \frac{|e\Phi|^3}{(r_+ - r_-)^2} e^{-|e\Phi|\beta} \approx S_n. \end{aligned} \tag{22}$$

It is seen that at $\rho_c = \rho_n$ this expression returns to the Hawking-Bekenstein formulae, as it has to, because, as we see, the thermal properties of the atmosphere, which were used to define the cutoff, do not depend on the scalar field charge e. It will be shown in the next section that the invariant distance in the Kerr metric is also independent of the atmosphere particle's angular momentum projection on the symmetry axis m.

For non-extreme black holes of mass greater than 10^{15}g it is a Klein-paradox process that dominates the charged pair production, and the emission rate is governed by a Schwinger-type formula (for pair production in a constant electric field). For black holes of smaller mass Hawking thermal process dominates[9].

2.2 Horizon Fluctuations

It was suggested in [2] that due to the atmosphere of high angular momentum particles, strong gravitational interactions occur near the Schwarzschild black hole horizon at an invariant distance of the order of $M^{1/3}$. This is because the total energy of the black hole's atmosphere fluctuates as any thermodynamical variable. Let us extend this idea to the case of the charged black holes.

According to the rules of thermodynamics, given the energy of the atmosphere $U(\rho)$ between the surfaces $r = r_+ + h(\rho)$ and $r = r_1$, and the number of particles $N(\rho)$ between these surfaces, the black hole's mass fluctuation between $r = 0$ and $r = r_+ + h(\rho)$ is

$$\Delta M(\rho) = \Delta U(\rho) \sim \frac{U(\rho)}{\sqrt{N(\rho)}}, \tag{23}$$

provided that Riemann tensor is sufficiently small near the horizon $\mathbf{R} \sim M^{-2} \ll 1$, so that we can apply the usual flat spacetime thermodynamic rules in the vicinity of the horizon. We have used the fact that the total energy M defined in (1) is fixed from the point of view of an external observer. The energy of the atmosphere above the surface $r = r_+ + h(\rho)$ is given by:

$$U(\rho) = \frac{\partial}{\partial \beta}(\beta F) \sim \frac{(r_+ - r_-)}{\rho^2}.$$

The number of particles is $N \sim S$. Therefore:

$$\Delta M(\rho) \sim \frac{(r_+ - r_-)}{r_+ \rho}.$$

Right now we are facing problem mentioned in the section 1: we had to use renormalized total energy U^{ren} instead of U. However, as far as only the variation of the total energy is needed, one may replace ΔU^{ren} by ΔU. Indeed, all divergent terms in the energy-momentum tensor are functions of the curvature tensor components and their derivatives only[8], so they are canceled from the expression for fluctuation of the total matter energy[2].

Let us estimate how the fluctuating mass gives rise to uncertainty in the location of the horizon. A point r' is outside the horizon if $r' - r_+(M(r')) > 0$, where $M(r')$ is the black hole energy between $r = 0$ and $r = r'$. Clearly, if

$$\Delta(r' - r_+(M(r'))) = \Delta(r_+(M(r')) > \; r' - r_+(M(r')) = h, \tag{24}$$

then the point r' is in the superposition of being inside and outside the horizon. Using eq. (2) we obtain:

$$\Delta r_+ = \frac{2\Delta M r_+}{r_+ - r_-} = \frac{1}{\rho}. \tag{25}$$

From eq. (21) we see that relation $\Delta r_+ < h$ holds if

$$\frac{1}{\rho} < \frac{\rho^2 (r_+ - r_-)}{r_+^2};$$

that is, the semi-classical approach is valid as long as

$$\rho > \rho_{min} = \left(\frac{r_+^2}{r_+ - r_-} \right)^{1/3}. \tag{26}$$

This equation implies that in Schwarzschild metric $\rho_{min} = M^{-1/3}$, see ref. [2]. In the case of extreme black hole ρ_{min} becomes infinite.

[2]In the ref. [13] S.Mukohyama and W.Israel argued that if one correctly identifies the ground state — the Boulware vacuum, then the total energy near the black hole horizon is finite. They use this fact to advocate the brick-wall model. Conclusions of this section are independent of choice of the ground state; the thermal fluctuations depend only on the number of modes near the horizon.

We have derived eq. (25) neglecting fluctuations of the atmosphere's total charge $\mathcal{Q}$. Nevertheless, it remains valid if take account of the $\mathcal{Q}$-fluctuation. Indeed,

$$\begin{aligned}\Delta r_+ &= \sqrt{\left(\frac{r_+}{r_+ - r_-}\right)^2 \Delta M^2 + \frac{Q^2}{(r_+ - r_-)^2}\Delta Q^2} \\ &= \frac{r_+}{r_+ - r_-}\sqrt{\Delta U^2 + \frac{Q^2}{r_+^2}\Delta \mathcal{Q}^2} = \frac{r_+}{r_+ - r_-}\Delta\left(U - \frac{Q}{r_+^2}\mathcal{Q}\right).\end{aligned}$$

Since the quantity in the curly brackets of the last equation is additive we estimate it's fluctuation as follows:

$$\Delta\left(U - \frac{Q}{r_+^2}\mathcal{Q}\right) \simeq \langle E - \frac{Q}{r_+^2}e\rangle\sqrt{N} \simeq T_H\frac{r_+}{\rho}.$$

Putting all this together leads to eq. (25).

2.3 Kerr Black Holes

Consider a rotating black hole with projection of angular momentum on the symmetry axis J (called simply black hole's "angular momentum"). The metric in Boyer-Lindquist coordinates is given by[6]:

$$\begin{aligned} ds^2 = \left(1 - \frac{2Mr}{\Sigma^2}\right)dt^2 - \frac{\Sigma^2}{D}dr^2 - \Sigma^2 d\theta^2 \\ - \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma^2}\sin^2\theta\right)\sin^2\theta d\varphi^2 + \frac{2\cdot 2Mra}{\Sigma^2}\sin^2\theta d\varphi dt. \end{aligned} \tag{27}$$

where $a = J/M$,

$$D = r^2 - 2Mr + a^2, \quad \Sigma^2 = r^2 + a^2\cos^2\theta. \tag{28}$$

The event horizon r_+ and surface r_- are defined as

$$r_{\substack{+\\-}} = M^+_-\sqrt{M^2 - a^2}. \tag{29}$$

There is a surface such that no static observer can exists inside it (frame dragging effect). Definition of this surface, called static limit, is

$$r_0(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}.$$

As in the case of the charged black hole there exists Hawking radiation with the temperature T_H given by [9]:

$$T_H = \frac{(r_+ - r_-)}{\mathcal{A}} = \frac{(r_+ - r_-)}{4\pi(r_+^2 + a^2)} = \frac{(r_+ - r_-)}{8\pi M r_+}.$$

Inertial observer at infinity measures flux of energy and angular momentum from the rotating black hole. We are interested to study what happens to the system of Hawking particles reflected back to the black hole by the space-time curvature.

Recall that in the Kerr metric one can separate variables in the wave equation (8)[9, 10]. The basis functions are:

$$u_{E\lambda_{lm}m}(x) = (r^2+a^2)^{-1/2} R_{lm}(r) S_{lm}(\cos\theta) e^{im\varphi} e^{-iEt},$$

where S_{lm} is the spherical harmonics with eigenvalue $\lambda_{lm}(aE)$, $l = 0, 1, 2 \ldots$, $m = -l, -l+1, \ldots l-1, l$.

The radial part $R_{lm}(r)$ satisfies the following equation:

$$[\frac{d^2}{dr^{*2}} - V_{Elm}(r)] R_{lm}(r) = 0,$$

where r^* and $V_{Elm}(r)$ are defined by the following relations:

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{D}, \tag{30}$$

$$\begin{aligned} V_{Elm}(r) \quad = \quad & -(E - m\frac{a}{r^2+a^2})^2 + \lambda_{lm}(aE)\frac{D}{(r^2+a^2)^2} + \\ & \frac{2(Mr-a^2)D}{(r^2+a^2)^3} + \frac{3a^2D^2}{(r^2+a^2)^4}. \end{aligned} \tag{31}$$

The centrifugal barrier is

$$V^c_{Elm}(r) = \lambda_{lm}(aE)\frac{D}{(r^2+a^2)^2}.$$

It is attractive for $r_+ \le r < r_1$ and repulsive for $r > r_1$, where r_1 is a physically acceptable root of equation $\frac{dV^c}{dr} = 0$:

$$r_1 = M + 2\sqrt{M^2 - \frac{1}{3}a^2} \cdot \cos\frac{\alpha}{3}, \tag{32}$$

where

$$\cos\alpha = \frac{1 - a^2/M^2}{(1 - a^2/3M^2)^{3/2}}.$$

Once again the tunneling through the angular momentum barrier may be neglected for all but the lowest angular momentum modes since $V^c_{lm} \sim \lambda_{lm} \sim l^2$.

The thermal atmosphere of the black hole extends from the horizon up to the surface $r = r_1$, where r_1 is the turning point of the centrifugal barrier. Study of two limit cases $a = 0$ and $a = M$ shows that $r_1 > r_0$ for any black hole. In other words the ergosphere lies under the external boundary of the atmosphere.

Let us calculate a number of wave equation solutions in the vicinity of the black hole. This number is equal to the twice of number of quantum states of the atmosphere, because of two-fold degeneration of energy levels (along the symmetry axis). In the WKB approximation:

$$\Gamma = 2(2\pi)^{-3} \sum_m \int d\varphi \int dr \int dp_r \int d\theta \int dp_\theta.$$

Here the covariant radial momentum p_r associated with the differential operator $\hat{p}_r$ in the wave equation is given by [6]:

$$p_r^2 = D^{-2}[(r^2+a^2)E - am]^2 - \lambda_{lm}D^{-1}, \tag{33}$$

and the polar momentum p_θ associated with $\hat{p}_\theta$:

$$p_\theta^2 = \lambda_{lm} - (aE\sin\theta - \frac{m}{\sin\theta})^2. \tag{34}$$

Introduce a cut-offs h and L which are defined as $\phi(x) = 0$ if $r \geq L$ and $r \leq r_+ + h(\theta)$. Now allow h to be some function of θ.

Let us change variables of integration replacing the pair (p_r, p_θ) by the pair of constants of motion (λ, E) (here $\lambda_{lm} \equiv \lambda$). Jacobian of this transformation is

$$\frac{\partial(p_r, p_\theta)}{\partial(\lambda, E)} = \frac{2D^{-2}(r_+^2+a^2)^2(E - m\Omega_H) - 2D^{-1}(aE\sin\theta - m/\sin\theta)a\sin\theta}{4p_r(\lambda,E)p_\theta(\lambda,E)}$$

Here $\Omega_H = a/2Mr_+$ is the angular velocity of the black hole[7]. Since the main contribution to the atmosphere's partition function arises from the region near the horizon ($r \approx r_+ + h$) we neglect the second term in the last equation and obtain (note eq. (11))

$$\begin{aligned}
\Gamma(E) &= 2(2\pi)^{-3}\sum_m \int d\varphi \int d\theta \int dr \int d\lambda \int dE \frac{\partial(p_r, p_\theta)}{\partial(\lambda, E)} \\
&\approx 2(2\pi)^{-3} 2\pi h \sum_m \int d\theta \int d\lambda \int dE \times \\
&\times \frac{D^{-2}(r_+^2+a^2)^2(E - m\Omega_H) + O(h^{-1})}{2\sqrt{2D^{-2}(r_+^2+a^2)^2(E-m\Omega_H)^2 - \lambda D^{-1}}\sqrt{\lambda - (aE\sin\theta - m/\sin\theta)^2}},
\end{aligned}$$

where D near the horizon is given by (11).

Now, the free energy for large β is

$$\begin{aligned}
F &= T_H \sum_m \int d\Gamma(E,m)\ln\left(1 - e^{(m\Omega_H - E)\beta}\right) \\
&\approx -\frac{h}{(2\pi)^2}\frac{T_H(r_+^2+a^2)^2}{D}\int d\theta \int_0^{\lambda_m} d\lambda \int_0^\infty dx \frac{xe^{-x\beta}}{\sqrt{(r_+^2+a^2)^2x^2 - \lambda D}} \times \\
&\times \int dm \frac{1}{\sqrt{\lambda - [a\sin\theta(x + m\Omega_H) - m/\sin\theta]^2}} + O(e^{-|m|\Omega_H\beta}) \\
&= -\frac{1}{2\pi}T_H(r_+^2+a^2)^3 \int d\theta \sin\theta \int_0^\infty dx x^2 e^{-x\beta} D^{-2} \\
&= -\frac{1}{\pi}T_H^4 \frac{(r_+^2+a^2)^3}{(r_+ - r_-)^2}\int d\theta \frac{\sin\theta}{h}
\end{aligned}$$

We have introduced a new variable of integration $x = E - m\Omega_H$ and taken account of the fact that contribution of particles with $m < 0$ is of the order of $O(e^{-|m|\Omega_H \beta})$ (like in the charged black hole case). The radial momentum p_r vanishes at $\lambda = \lambda_m$.

It is easy to convince, that the entropy is given by

$$S = \frac{4}{\pi} T_H^3 \frac{(r_+^2 + a^2)^3}{(r_+ - r_-)^2} \int_0^\pi d\theta \frac{\sin\theta}{h}. \tag{35}$$

This formula is consistent with eq. (19). (To see this put $Q = 0$ in eq. (19) and $a = 0$ in eq. (35)and compare).

Let us define invariant distance ρ as in eq. (21):

$$\rho = \sqrt{\frac{r_+^2 + a^2\cos^2\theta}{r_+ - r_-}} \sqrt{h(\theta)}. \tag{36}$$

Substitution to (35) gives

$$S = \frac{4}{\pi(4\pi)^3}(r_+ - r_-) \int_0^\pi \frac{d\theta \sin\theta}{h} = \frac{4}{\pi(4\pi)^3} \int_{-1}^1 \frac{d\xi(r_+^2 + a^2\xi^2)}{\rho^2(\xi)}.$$

We may choose $h(\theta)$ in such a way that this equation will coincide with the Hawking- Bekenstein expression for the black-hole entropy. For example, take $h(M, a|\theta) = \bar{h}(M, a)/(\cos^{-2}\theta + 3)$. The problem is that there are infinitely many ways to do this, and thus, one cannot fix the θ dependence of h. Despite this difficulty, however, we see that whatever function $h(\theta)$ that matches the Hawking-Bekenstein expression one chooses, the invariant cutoff ρ does not depend on the black hole parameters M and J and on the atmosphere's thermal properties. It is characterized by the event horizon only.

The fact that we do not know the exact form of the function $h(\theta)$ does not matter, if we are interested only in ratios of such thermodynamical variables as the total energy, entropy etc. In this case we need not know this function. This is the case if we are going to calculate the horizon fluctuations. We repeat the by now familiar procedure of sec. 2.2 and arrive at the formula (26) with r_+ and r_- defined by eq. (29). Note that deriving this result we have effectively averaged thermodynamical variables over all possible choices of h which yield the Bekenstein-Hawking entropy.

It is straightforward to generalize this result to the Kerr-Newman black hole. We will refer to eq. (26) as such generalization with

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{37}$$

It should be emphasized, that despite the fact that the ρ_{min} becomes infinite for the extreme black holes, the condition $h \ll M$ under which all calculations were done still holds. Indeed, using eq. (36) this condition reads

$$\rho \ll \frac{M^{3/2}}{(M^2 - a^2 - Q^2)^{1/4}}. \tag{38}$$

It is seen by substituting ρ_{min} from eq. (26) into eq. (38) that the ρ_{min} always matches (38).

3 Shock Wave Model

In this section we shall show that strong gravitational interaction in the vicinity of the charged black hole occurs at ρ given by eq. (26) by studying the gravitational interactions between an in-falling particle and the thermal atmosphere.

It is shown in the Appendix, that propagation of a massless particle on the charged black hole background can be described by the shock wave of a special form, so that the resulting metric is given by eqs. (48),(57),(58) and (60). Consider some massless particle falling into the black hole. The atmosphere (Hawking) particles interact with the shock wave generated by this particle. What is the probability that the state of the atmosphere is the same after the interaction took place?

At first, we shall study interaction of the in-falling particle with the one test atmosphere particle. Note that the Reissner-Nordstrøm geometry (1) takes the same approximate form as the Schwarzschild one near the horizon:

$$ds^2 = \frac{\rho^2(r_+ - r_-)^2}{4r_+^4}dt^2 - d\rho^2 - dx^2 - dy^2,$$

provided that we study a region with transverse distances much smaller than M. Here ρ is defined in (21) independently of the black hole charge. Denote

$$\tilde{M} \equiv \frac{r_+^2}{2(r_+ - r_-)},$$

then this metric reads

$$ds^2 = \frac{\rho^2}{(4\tilde{M})^2}dt^2 - d\rho^2 - dx^2 - dy^2. \tag{39}$$

Parameter $\tilde{M}$ plays for metric of the charged black hole the same role as the mass M for the neutral one.

We now follow the arguments of ref. [2] for the Schwarzschild black hole. Define Rindler coordinates

$$u = T + z = \rho e^{t/4\tilde{M}}, \quad v = T - z = -\rho e^{-t/4\tilde{M}}. \tag{40}$$

Then metric (39) is simply Minkowski space in these coordinates

$$ds^2 = dudv - dx^2 - dy^2. \tag{41}$$

Let k_μ, p_μ be the momenta of in-going and atmosphere (Hawking) particles respectively. The gravitational field of massless point-like particle in Minkowski space is described by the line element [12]

$$ds^2 = du\left[dv + 2k^v \ln\left(\frac{\tilde{x}^2}{M^2}\right)\delta(u - u_0)du\right] - dx^2 - dy^2, \tag{42}$$

where $\tilde{x}^2 = x^2 + y^2$, $u = T + z$ and $v = T - z$. The massless particle moves in the v direction with constant u_0 and momentum k^v.

The effect of the shock wave on the massless particle propagating in the metric (42) with initial momentum p_μ is a discontinuity in the v direction at $u = u_0$:

$$\Delta v = -2k^v \ln\left(\frac{\tilde{x}^2}{M^2}\right), \tag{43}$$

and a refraction in the transverse direction:

$$p_x(u) - p_x = \frac{4k^v}{\tilde{x}^2} x p_v \theta(u - u_0), \tag{44}$$

and similarly for $p_y(u)$.

By forming wave packets describing a high angular momentum Hawking particle before and after crossing the shock wave, and then calculating their scalar product one finds that the probability to be in the same state after crossing the shock wave is[2]

$$P_1 \sim 1 - \frac{\tilde{M}^2\epsilon^2}{\rho^4},$$

where ϵ is the energy of the atmosphere particle. Also, the probability for one particle in the atmosphere to have changed angular momentum is

$$P_{\Delta l \neq 0} = \frac{\tilde{M}^2\epsilon^2}{\rho^4}.$$

This is the result of interaction of in-falling particle with one atmosphere particle.

The number of particles which are affected by the shock wave of the in-going particle when it reaches ρ can be deduced from eqs. (19,21):

$$N(\rho) \sim S(\rho) \sim \frac{r_+^2}{\rho^2}.$$

Thus the probability for the atmosphere above ρ to remain in the initial state is

$$P_{tot} = P_1^{N(\rho)} = \left(1 - \frac{\tilde{M}^2\epsilon^2}{\rho^4}\right)^{N(\rho)} \sim e^{-\tilde{M}^2\epsilon^2 r_+^2/\rho^6}.$$

We see that the state of the atmosphere is changed when the particle reaches $\rho = (\tilde{M} r_+ \epsilon)^{1/3}$.

The minimal ϵ one can consider is $\frac{1}{r_+}$ since otherwise the wavelength of the in-going particle is larger than the radius of the black hole. Thus the minimal ρ is given by the following eq.

$$\rho_{min} = \tilde{M}^{1/3} \sim \left(\frac{r_+^2}{r_+ - r_-}\right)^{1/3}.$$

which coincides with eq. (26).

It can be easily shown using the same arguments as in ref. [2] that the information carried by an in-going massless spin-less charged particle is encoded in the state of the atmosphere when the particle reaches ρ_{min}.

4 Discussion

We saw in the previous sections that the high angular momentum Hawking particles get reflected by the centrifugal barrier near the black hole horizon. It turns out that the number, total energy and other statistical quantities of these particles, which are found in the region over the horizon, depend only on the black hole parameters. This allows to speak about the atmosphere of Hawking particles near the horizon, as about pure quantum geometrical phenomenon.

Since the Hawking radiation has a Planckian structure it is natural to say that a system consisting of a black hole and its radiation is in the thermal equilibrium state. One is interested to know thermodynamical parameters of the atmosphere in this state. In order to calculate entropy and the free energy of the atmosphere we counted the number of modes of the scalar field near the horizon, and then used well-known statistical thermodynamic formulae. Both entropy and the free energy are proved to be divergent at the horizon.

In order to deal with these divergences we introduced a cutoff at the surface $r = r_+ + h$. The leading order term $O(h^{-1})$ in the expression for the entropy is the contribution of the horizon to the total atmosphere entropy. The value of h was fixed by the requirement that the black hole entropy be a quarter of the horizon area, according to the Hawking-Bekenstein formula. This program was carried out for the Schwarzschild black hole in ref. [4], and for charged and rotating black holes in sections 2.1 and 2.3 respectively. We found that the value of the cutoff expressed in terms of the invariant distance is the same for all black holes independently of their mass, charge and angular momentum and is of order unity. At this (Planck) scale the semi-classical approach breaks down.

Up to now the back reaction of the emerging radiation was neglected. Accounting for the back reaction gives rise to the black hole's mass decreasing and to the black hole's parameters fluctuation. The first phenomenon can be neglected at short time scales, provided that the black hole's energy is much larger than the Planck mass. The second one is responsible for the strong gravitational interactions occurring near the horizon.

Indeed, by studying the atmosphere's mass fluctuations and interactions between Hawking and in-coming particles, we showed that the classical trajectories near the horizon cease to exist at invariant distances smaller than $\rho_{min} = [r_+^2/(r_+ - r_-)]^{1/3}$. This means that one cannot continue to use a non-quantized background metric at this scale, the semi-classical approach becomes invalid.

If the black hole has small charge and angular momentum then $\rho_{min} \ll M$ and the question is: how does the high frequency part of the Hawking radiation spectrum affected by the gravitational interactions. Our analysis does not provide the analytical answer to this question. However, the fact that information is encoded out of the horizon, may be the reason *pro* the possibility of S-matrix construction advocated by 't Hooft.

An essentially new effect arises, when we turn our attention to the extreme black holes. As the distance between horizon r_+ and the surface r_- becomes smaller, the strong gravitational interactions occur at greater invariant distance, the smaller frequency Hawking quanta are affected, and so the whole spectrum becomes non-thermal. This in turn implies that in the case of the extreme black hole the information of an in-going particle is encoded at distances much

larger than black hole's radius. In this case, the semi-classical approach breaks down at all scales.

Despite the fact that the minimum invariant distance from the horizon ρ_{min} at which the semi-classical theory still holds becomes infinite for the extreme black holes, its counterpart in the Boyer-Lindquist coordinates $h_{min} \equiv h(\rho_{min})$ vanishes. However it still remains greater than value of the cutoff. Indeed, definition of the invariant distance implies that in general

$$h_{min} \sim \rho_{min}^{-1} \sim \frac{(r_+ - r_-)^{1/3}}{M^{2/3}}.$$

The cutoff in these coordinates is

$$h_{\text{cutoff}} \sim \frac{(r_+ - r_-)}{M^2}$$

So, $h_{\text{cutoff}} < h_{min}$ for all black holes. Thus, one cannot reach the extreme black hole horizon (naked singularity) without entering a region below the h_{min} which cannot be investigated with the semi-classical theory. We believe that the gravitational interactions which occur in this region somehow prohibit the particle to reach the horizon (and to violate causality).

There is another simple argument which explains why one cannot reach the extreme black hole horizon in spite the fact that the non-invariant cutoff vanishes. The proper time it takes a freely falling observer to reach the horizon from some point R out of the horizon is infinite because this time is just the invariant distance

$$\Delta\tau = \rho \sim \frac{M}{r_+ - r_-}\sqrt{2M - R},$$

which is infinite for all R in our case.

As we explained, the semi-classical approach is wrong if the radial coordinate in the Boyer-Lindquist coordinates is smaller than $r_+ + h_{min}$. This leads to some inconsistency when we use the brick wall model, since we count semi-classical modes between $r_+ + h_{\text{cutoff}}$ and $r_+ + h_{min}$ also. However, we can neglect the contribution of this region, since the number of modes in the atmosphere is determined by the phase space volume in the nearest vicinity of the cutoff. $\Gamma(\rho_{min})$ is of order $O((M(r_+ - r_-))^{2/3})$ which is negligible compared to the leading order term $O(M^2)$. Moreover, the explicit (quantum gravitational) accounting for the processes occurring in this region should not change the estimate of ρ_{min}, since a completely different approach,— shock wave model, which does not use the counting of states, gives the same result. It is interesting to note that in the extreme black hole case the problematic region shrinks and has zero measure.

We did not discuss so far how an external observer learns about the black hole's horizon fluctuations. We noted, that the black hole mass is fixed in his frame (apart of small decrease caused by the back reaction). So, in order to obtain information about the horizon fluctuations, the external observer should study trajectories of test particles which pass near the horizon. The fluctuating horizon may trap, with some probability, a particle with definite quantum numbers. And it may release another particle, which, in general, will not bear the same quantum numbers, since we have no information about the state of order or disorder of matter inside the black hole.

Therefore, the out-going geodesics which pass near the horizon will be thermally averaged and will differ from the classical geodesics.

The external observer may also ask another observer who is found near the horizon, to send him information about the atmosphere's state. Then he must process this information bearing in mind that the Hawking temperature of the observer in the atmosphere is blue-shifted, and the light signals are red-shifted compared to those in his frame. It may be interesting to investigate this issue in more detail.

The fact that the Hawking radiation is merely fluctuations of vacuum, may put in doubt our approach to the atmosphere as a system of real on-shell particles. As was pointed out in ref. [2], it is expected to be a valid approximation if the S-matrix ansatz of 't Hooft is correct.

In summary, our analysis suggests that strong gravitational interactions occur near the black hole horizon at an invariant distance of order $\rho_{min} = [r_+^2/(r_+ - r_-)]^{1/3}$. At smaller distances the semi-classical approach breaks down.

A Appendix. Shock wave on charge black hole background

In the Appendix we shall show that the shock wave of a special form (48), generated by a point-like massless particle can propagate on the charged non-extreme black hole background.

Recall the general result obtained by Dray and 't Hooft[12]: Given a solution of the vacuum Einstein equations of the form

$$ds^2 = 2A(u,v)dudv + g(u,v)h_{ij}(x^i)dx^idx^j; \tag{45}$$

if the following conditions hold

$$A_{,v}|_{u=0} = 0 = g_{,v}|_{u=0}, \tag{46}$$

$$\frac{A}{g}\Delta f - \frac{g_{,uv}}{g}f = 32\pi p A^2 \delta(\tilde{x}), \tag{47}$$

where $f = f(x^i)$ represents the shift in v, Δf is the Laplacian of f with respect to the 2-metric h_{ij} and $\tilde{x}$ is the transverse distance, then the shift in v at $u = 0$ can be introduced so that the resulting space-time solves the field equation with a photon at the origin $\tilde{x} = 0$ of the (x^i) 2-surface and $u = 0$. The resulting metric is then described as follows:

$$d\hat{s}^2 = 2A(u, v+\theta f)du(dv + \theta f_{,i}dx^i) + g(u, v+\theta f)h_{ij}dx^idx^j. \tag{48}$$

Let us verify whether conditions of this statement are satisfied in the case we are interested in. At first, we introduce new coordinates $\tilde{u}, \tilde{v}$ which are labels for outgoing and in-going, radial, null geodesics[7]. The geodesics are solutions of the following equation:

$$ds^2 = 0 = \frac{D}{r^2}dt^2 - \frac{r^2}{D}dr^2. \tag{49}$$

Hence they are given by:

$$\tilde{u} = t - r^*, \quad \tilde{v} = t + r^*, \tag{50}$$

where r^* is defined by (5) and reads as follows

$$r^* = r + M \ln \frac{D}{D_0} + \frac{2M^2 - Q^2}{(r_+ - r_-)} \ln \frac{r - r_+}{r - r_-}, \tag{51}$$

and D_0 is integration constant.

The line element (1) in terms of new coordinates reads:

$$ds^2 = \frac{D}{r^2} d\tilde{v} d\tilde{u} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{52}$$

It has obvious pathology at the horizon: $D(r)|_{r=r_+} = 0$. In order to remove it we change coordinates again[7]. Using eqs (50),(51) one gets

$$e^{(\tilde{v}-\tilde{u})/(4M)} = e^{r^*/2M} = e^{r/2M} \left(\frac{r - r_+}{2M} \right)^{\frac{1}{2}+\alpha} (r - r_-)^{\frac{1}{2}-\alpha}, \tag{53}$$

where

$$\alpha = \frac{2M^2 - Q^2}{2M(r_+ - r_-)},$$

and D_0 is fixed using requirement that equation (53) must yield the correct result at $Q = 0$. Thus, new coordinates are defined as follows

$$u = -e^{-\tilde{u}\beta/4M}; \quad v = e^{\tilde{v}\beta/4M}, \tag{54}$$

where β is a certain constant to be fixed in such a way as to remove the pathology in the metric (52). We have

$$\frac{D}{r^2} d\tilde{u} d\tilde{v} = \frac{16M^2 (2M)^{(\frac{1}{2}+\alpha)\beta}}{\beta^2 r^2} e^{-\frac{\beta r}{2M}} (r - r_+)^{1-(\frac{1}{2}+\alpha)\beta} (r - r_-)^{1-(\frac{1}{2}-\alpha)\beta} du dv,$$

where eqs. (53) and (54) were used. Clearly, the value of β we need is

$$\beta = \beta_0 \equiv \frac{1}{\frac{1}{2} + \alpha}. \tag{55}$$

Thus, by substitution this into (52) the line element reads

$$ds^2 = \frac{32M^3}{\beta_0^2 r^2} e^{-\frac{\beta_0 r}{2M}} (r - r_-)^{1-(\frac{1}{2}-\alpha)\beta_0} du dv - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{56}$$

In coordinates u, v, θ, and φ metric of the charged black hole (56) has no pathologies. The only singularity at $r = 0$ is a physical one.

Functions A and g defined in eq. (45) are

$$A = \frac{16M^3}{\beta_0^2 r^2} e^{-\frac{\beta_0 r}{2M}} (r - r_-)^{1-(\frac{1}{2}-\alpha)\beta_0}, \tag{57}$$

$$g = -r^2. \tag{58}$$

Note the useful relation following from eqs. (53) and (54):

$$uv = -e^{\frac{\beta_0 r}{2M}} \frac{r - r_+}{2M} (r - r_-)^{(\frac{1}{2} - \alpha)\beta_0}.$$

It shows that A and g are functions of the product $u \cdot v$, so

$$A(u \cdot v)_{,v} \propto u; \quad g(u \cdot v)_{,v} \propto u,$$

and therefore conditions (46) are satisfied.

Also,

$$g_{,u} = -\frac{4MD}{\beta_0 r} e^{-\frac{\beta_0 r}{2M}} \frac{2M}{r - r_+} (r - r_-)^{(\alpha - \frac{1}{2})\beta_0} v,$$

$$g_{,uv} = \frac{A}{r} \left[2(r - M) - D(\frac{1}{r} + \frac{\beta_0}{2M}) - (r - r_-) + (r - r_+)(\alpha - \frac{1}{2}) + \frac{\beta_0 r^2}{2M} \right].$$

Following ref. [12] we arrange the coordinates so that the massless particle is at $\theta = 0 = u$. Then, upon substitution of these relations into equation (47) one obtains

$$\Delta f - \lambda(M, Q) f = -2\pi\kappa(M, Q)\delta(\theta), \tag{59}$$

where

$$\lambda(M, Q) = \frac{1}{r_+} \left[2(r_+ - M) - (r_+ - r_-) + \frac{\beta_0 r_+^2}{2M} \right],$$

$$\kappa(M, Q) = 2^8 p \frac{M^3}{\beta_0^2} (r_+ - r_-)^{1 + (\alpha - \frac{1}{2})\beta_0} e^{-\frac{\beta_0 r}{2M}}.$$

Note that $\lambda(M, 0) = 1$, and $\kappa(M, 0) = 2^9 p e^{-1} M^4$. This is the result obtained by Dray and 't Hooft.

We can find solution to eq. (59) by expanding f in terms of spherical harmonics. However, since only spherical harmonics with $m = 0$ contribute, it is suffice to expand f in terms of Legendre polynomials $P_l(\cos\theta)$. Then

$$f = \kappa \sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l+1) + \lambda} P_l(\cos\theta). \tag{60}$$

Since the asymptotic behavior of the Legendre polynomials at large l is

$$P_l \sim \frac{1}{l^{1/2}},$$

this sum converges for all λ such that $0 < \lambda \leq 1$, i.e. for non-extreme black holes. $\lambda = 0$ corresponds to the extreme black holes $|Q| = M$; in this case $\kappa(M, M) = 2^8 p M^3$ and f diverges according to the following law:

$$f^{\text{ext}} = \frac{\kappa(M, M)}{2\lambda(M, M)} \sim \frac{1}{\sqrt{M - |Q|}}.$$

Acknowledgments

The author would like to thank N.Itzhaki, F.Englert and especially A.Casher for fruitful discussions on related topics.

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